Bernstein-type inequalities for rational functions in weighted Bergman spaces

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Abstract

We prove Bernstein-type inequalities in weighted Bergman spaces of the unit disc $\mathbb{D}$, for rational functions in $\mathbb{D}$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$. The asymptotic sharpness of each of these inequalities is shown as $n \to \infty$ and $r \to 1$. Our results extend a result of K. Dyakonov who studied Bernstein-type inequalities (for the same class of rational functions) in the standard Hardy spaces.

1. Introduction

Estimates of the norms of derivatives for polynomials and rational functions (in different functional spaces) is a classical topic of complex analysis (see surveys given by A. A. Gonchar [Go], V. N. Rusak [Ru] and Chapter 7 of [BoEr]). Here, we present such inequalities for rational functions f of degree n with poles in $\{z : |z| > 1\}$, involving Hardy norms and weighted-Bergman norms. Some of these inequalities are applied in many domains of analysis: for example 1) in matrix analysis and in operator theory (see "Kreiss Matrix Theorem" [LeTr, Sp] or [Z1, Z5] for resolvent estimates of power bounded matrices), 2) to "inverse theorems of rational approximation" using the *classical Bernstein decomposition* (see [Da, Pel, Pek]), but also 3) to effective Nevanlinna-Pick interpolation problems (see [Z3, Z4]). Let $\mathcal{P}_n$ be the complex space of polynomials of degree less or equal than $n \geq 1$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and $\overline{\mathbb{D}} =$

$\{z \in \mathbb{C} : |z| \leq 1\}$ its closure. Given $r \in [0,\, 1)$, we define

$$\mathcal{R}_{n,\,r} = \left\{ \frac{p}{q} \, : \, p,\, q \in \mathcal{P}_n,\ \mathrm{d}^\circ p < \mathrm{d}^\circ q,\ q(\zeta) = 0 \Longrightarrow \zeta \notin \frac{1}{r}\mathbb{D} \right\},$$

(where $\mathrm{d}^\circ p$ means the degree of any $p \in \mathcal{P}_n$), the set of all rational functions in $\mathbb{D}$ of degree less or equal than $n \geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $r = 0$, we get $\mathcal{R}_{n,\,0} = \mathcal{P}_{n-1}$.

1.1. Definitions of Hardy spaces and radial weighted Bergman spaces

a. The standard Hardy spaces $H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$:

$$H^p = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k \, : \, \|f\|_{H^p}^p = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^p \, \mathrm{d}m(z) < \infty \right\}.$$

b. The standard radial weighted Bergman spaces are denoted by $L^p_a\,(w)$, $1 \leq p < \infty$ (where "a" means analytic),

$$L^p_a\,(w) = \left\{ f \in \mathrm{Hol}\,(\mathbb{D}) : \, \|f\|^p_{L^p_a(w)} = \int_0^1 w\,(\rho) \int_{\mathbb{T}} |f\,(\rho\zeta)|^p \, \mathrm{d}m(\zeta)\mathrm{d}\rho < \infty \right\},$$

where $\mathrm{Hol}\,(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$, $w \geq 0$, $\int_0^1 w\,(\rho)\,\mathrm{d}\rho < \infty$, and m stands for the normalized Lebesgue measure on $\mathbb{T}$. Classical power-like weights correspond to $w(\rho) = w_\beta(\rho) = \left(1 - \rho^2\right)^\beta \rho$ for $\beta > -1$, where $L^p_a\,(w_\beta) = L^p_a\left(\left(1 - |z|^2\right)^\beta \mathrm{d}x\mathrm{d}y\right)$. For general properties of these spaces we refer to [HedKorZhu, Zhu].

1.2. Statement of the problem and the result

Generally speaking, given a Banach space X of holomorphic functions in $\mathbb{D}$, we are searching for the "best possible" constant $\mathcal{C}_{n,\,r}(X)$ such that

$$\|f'\|_X \leq \mathcal{C}_{n,\,r}(X)\, \|f\|_X\,,$$

$\forall f \in \mathcal{R}_{n,\,r}$.

Throughout this paper the letter c denotes a positive constant that may change from one step to another. For two positive functions a and b, we say that a is dominated by b, denoted by $a = O(b)$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a = O(b)$ and $b = O(a)$ hold.

If X is a Hardy space (see [Dy1, Dy2] and Subsection 1.3 below) then

$$\mathcal{C}_{n,\,r}\left(X\right) \asymp \frac{n}{1-r}, \tag{$\star$}$$

for all $n \geq 1$ and $r \in [0,\,1)$. Our result is that the above estimate ($\star$) is still valid if X is the radial weighted Bergman space $L^p_a\left(w\right)$, $1 \leq p < \infty$ with $w = w_\beta$, $\beta > -1$, or -more generally- whith "reasonably" decreasing weights w, where "reasonably" means "not too fast", (see 1.1.b above for the definition of this space). More precisely, we prove (in Section 2 below) the following theorem.

Theorem. *(1) Radial weighted Bergman spaces. Let $1 \leq p < \infty$ and w be an integrable nonegative fonction on $(0,\,1)$. We have*

$$\mathcal{C}_{n,\,r}\left(L^p_a\left(w\right)\right) \leq K\frac{n}{1-r}, \tag{1}$$

where K is a postive constant depending only on p.

(2) Some specific weights. Let $1 \leq p < \infty$ and w be an integrable nonegative fonction on $(0,\,1)$ such that $\rho \mapsto (1-\rho)^{-\gamma}w(\rho)$ is increasing on $[r_0,\,1)$ for some $\gamma > 0$, $0 \leq r_0 < 1$. There exists a positive constant K' depending only on w and p such that

$$K'\frac{n}{1-r} \leq \mathcal{C}_{n,\,r}\left(L^p_a\left(w\right)\right) \leq K\frac{n}{1-r}, \tag{2}$$

where K is defined in (1) and where the left-hand side inequality of (2) holds for $n > \left[\frac{\gamma+2}{p}\right] + 1$, $r \in [r_0,\,1)$, $[x]$ meaning the integer part of any $x \geq 0$. In particular, (2) holds for classical power-like weights $w(\rho) = w_\beta(\rho) = \left(1-\rho^2\right)^\beta \rho$ for $\beta > -1$,

1.3. Known result: an estimate for $\mathcal{C}_{n,\,r}\left(H^p\right)$, $1 \leq p \leq \infty$

From now on, if $\sigma \subset \mathbb{D}$ is a finite subset of the unit disc ($\operatorname{card}\sigma = n$), then

$$B_\sigma = \prod_{\lambda\in\sigma} b_\lambda$$

is the corresponding finite Blaschke product (say of order n), where $b_\lambda = \frac{\lambda - z}{1-\overline{\lambda} z}$, $\lambda \in \mathbb{D}$.

The first known result is a special case of a K. Dyakonov' s theorem [Dy2, Theorem 1] (the case where φ is an inner function belonging to the Sobolev space W^s_∞ or the Besov space B^s_∞ for some $s > 0$, that is to say a finite Blaschke product (say with n zeros inside $r\overline{\mathbb{D}}$, $r < 1$)): let $p \in [1, +\infty]$, we have

$$\mathcal{C}_{n,r}\left(H^p\right) \leq c_p \frac{n}{1-r}, \tag{4}$$

where c_p is a constant depending only on p. More precisely as regards inequality (4), the case $p \in (1, +\infty)$ is treated in [Dy2, Theorem 1], the case $p = 1$, in [Dy2, Corollary 1] and the case $p = +\infty$ is given in [BoEr, Theorem 7.1.7].

The second result is also due to K. Dyakonov [Dy1, Theorem 1' - page 373] and was originally proved for the Hardy spaces of the upper half plane $\mathbb{C}_+$, $H^p\left(\mathbb{C}_+\right)$. Let us recall it.

Theorem. 1'. *Let $1 < p < \infty$, θ be an inner function and $K^p_\theta = H^p\left(\mathbb{C}_+\right) \cap \theta \overline{H^p\left(\mathbb{C}_+\right)}$ be the corresponding "model" or "star-invariant" subspace. The operator $\frac{\mathrm{d}}{\mathrm{d}x} : f \mapsto f'$ acts boundedly from K^p_θ to $L^p = L^p\left(\mathbb{R}\right)$ if and only if $\theta' \in L^\infty$. Moreover,*

$$A_p \left\|\theta'\right\|_\infty \leq \left\|\frac{\mathrm{d}}{\mathrm{d}x}\right\|_{K^p_\theta \to L^p} \leq B_p \left\|\theta'\right\|_\infty, \tag{5}$$

for some constants $A_p > 0$ and $B_p > 0$.

The techniques of K. Dyakonov applied in order to prove (5) in [Dy1], give an analog of Theorem 1' for the Hardy spaces $H^p = H^p(\mathbb{D})$ of the unit disc $\mathbb{D}$. This analog would be (θ must be a finite Blaschke product (say as before with n zeros inside $r\overline{\mathbb{D}}$, $r < 1$) since we want the differentiation operator to be bounded)):

$$A^{'}_p \frac{n}{1-r} \leq \mathcal{C}_{n,r}\left(H^p\right) \leq B^{'}_p \frac{n}{1-r}, \tag{6}$$

for some constants $A^{'}_p > 0$ and $B^{'}_p > 0$, for every $p \in (1, \infty)$. In fact it is easily verified that (6) is also valid for $p = 1, \infty$ (using (4) for the right-hand side inequality and the test function $B = b^n_r$ so as to prove the left-hand side one).

For the special case $p=2$, it has been proved later in [Z2] that there exists a limit

$$\lim_{n\to\infty}\frac{\mathcal{C}_{n,\,r}\left(H^2\right)}{n}=\frac{1+r}{1-r},\tag{7}$$

for every $r,\ 0\le r<1$.

Our Theorem above in Subsection 1.1 is an extension of the K. Dyakonov's result (6) to radial weighted Bergman spaces $L^p_a\left(w\right),\ 1\le p<\infty$, for $w=w_\beta,\ \beta>-1$, or -more generally- when w is "reasonably" decreasing to 0, that is too say not fast. We prove it in Section 2 below. In Section 3, we discuss the validity of this Theorem for more general radial weights $w=w(\rho)$.

2. Proof of the theorem

We first prove the statement (1) of our theorem.

Proof of statement (1) of the theorem (the upper bound). First, we notice that

$$\|f\|_{L^p_a(w)}\asymp\frac{1}{\pi}\int_{C_\alpha}|f(w)|^p\,w\left(\rho\right)\mathrm{d}x\mathrm{d}y\tag{8}$$

for all $f\in L^p_a(w)$, where $C_\alpha=\left\{z\,:\,\alpha<|z|<1\right\}$, for any $0\le\alpha<1$. Let $f\in\mathcal{R}_{n,\,r}$ with $r\in[0,\,1)$ and $n\ge1$. Let also $\rho\in(0,\,1)$ and $f_\rho\,:\,w\mapsto f(\rho w)$. Using (8) with $\alpha=\frac{1}{2}$ we get

$$\|f'\|^p_{L^p_a(w)}\asymp\frac{1}{\pi}\int_{C}|f'(w)|^p\,w\left(\rho\right)\mathrm{d}x\mathrm{d}y=$$

$$=2\int_{\frac{1}{2}}^1 w\left(\rho\right)\left(\int_{\mathbb{T}}\left|f'_\rho(\zeta)\right|^p\mathrm{d}m(\zeta)\right)\mathrm{d}\rho=$$

$$=2\int_{\frac{1}{2}}^1 w\left(\rho\right)\left(\|f'_\rho\|^p_{H^p}\right)\mathrm{d}\rho=2\int_{\frac{1}{2}}^1 w\left(\rho\right)\frac{1}{\rho^p}\left(\left\|\left(f_\rho\right)'\right\|^p_{H^p}\right)\mathrm{d}\rho.$$

Now using the fact $f_\rho\in\mathcal{R}_{n,\,\rho r}\subset\mathcal{R}_{n,\,r}$ for every $\rho\in(0,\,1)$, we get

$$\int_{\frac{1}{2}}^1 w\left(\rho\right)\frac{1}{\rho^p}\left(\left\|\left(f_\rho\right)'\right\|^p_{H^p}\right)\mathrm{d}\rho\le$$

$$\leq 2^p \int_{\frac{1}{2}}^{1} w(\rho) \left(\mathcal{C}_{n,r}(H^p) \|f_\rho\|_{H^p}\right)^p \mathrm{d}\rho =$$

$$= (2\mathcal{C}_{n,r}(H^p))^p \int_{\frac{1}{2}}^{1} w(\rho) \int_{\mathbb{T}} |f_\rho(\zeta)|^p \,\mathrm{d}m(\zeta)\mathrm{d}\rho =$$

$$= (2\mathcal{C}_{n,r}(H^p))^p \int_C |f(w)|^p w(\rho)\,\mathrm{d}x\mathrm{d}y \asymp (\mathcal{C}_{n,r}(H^p))^p \|f\|^p_{L^p_a(w)}.$$

In particular, using the right-hand inequality of (4), we get

$$\mathcal{C}_{n,r}(L^p_a(w)) \leq K_p \frac{n}{1-r},$$

for all $p \in [1, \infty)$, and $\beta \in (-1, \infty)$, where K_p is a constant depending on p only.

□

It remains to prove the statement (2) of our theorem. To this aim, we first give two lemmas.

Lemma 1. *Let $r \in [0, 1)$ and $t \geq 0$. We set*

$$I(t, r) = \int_{\mathbb{T}} |1 - r\zeta|^{-t} \,\mathrm{d}m(\zeta) \textit{ and } \varphi_r(t) = \int_{\mathbb{T}} |1 + rz|^t \, dz$$

Then,

$$I(t, r) = \frac{1}{(1-r^2)^{t-1}} \varphi_r(t-2),$$

for every $t \geq 2$, where $t \mapsto \varphi_r(t)$ is an increasing function on $[0, +\infty)$ for every $r \in [0, 1)$. Moreover, both

$$r \mapsto \varphi_r(t-2) \textit{ and } r \mapsto I(t, r),$$

are increasing on $[0, 1)$, for all $t \geq 0$.

Proof. Indeed supposing that $t \geq 2$, we can write

$$I(t, r) = \frac{1}{1-r^2} \int_{\mathbb{T}} |b_r'| \frac{1}{|1-rz|^{t-2}} \mathrm{d}z,$$

(where $b_r = \frac{r-z}{1-rz}$), that is to say - using the changing of variable $\circ b_r$ in the above integral -

$$I(t,\, r) = \frac{1}{1-r^2}\int_{\mathbb{T}} |b_r'| \frac{1}{|1-rb_r\circ b_r(z)|^{t-2}}\mathrm{d}z =$$

$$= \frac{1}{1-r^2}\int_{\mathbb{T}} \frac{1}{|1-rb_r(z)|^{t-2}}\mathrm{d}z.$$

Now, $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$, which gives

$$I(t,\, r) = \frac{1}{(1-r^2)^{t-1}}\int_{\mathbb{T}} |1-rz|^{t-2}\,\mathrm{d}z,$$

or

$$I(t,\, r) = \frac{1}{(1-r^2)^{t-1}}\varphi_r(t-2). \tag{9}$$

We can write

$$\varphi_r(t) = \int_0^{2\pi} \left(1+r^2-2r\cos s\right)^{\frac{t}{2}}\mathrm{d}s =$$

$$= \int_0^{2\pi} \exp\left(\frac{t}{2}\ln\left(1+r^2-2r\cos s\right)\right)\mathrm{d}s.$$

Then

$$\varphi_r'(t) = \frac{1}{4}\int_0^{2\pi} \ln\left(1+r^2+2r\cos s\right)\exp\left(\frac{t}{2}\ln\left(1+r^2+2r\cos s\right)\right)\mathrm{d}s,$$

and

$$\varphi_r''(t) =$$

$$= \frac{1}{4}\int_0^{2\pi} \left[\ln\left(1+r^2-2r\cos s\right)\right]^2\exp\left(\frac{t}{2}\ln\left(1+r^2-2r\cos s\right)\right)\mathrm{d}s \geq 0,$$

for every $t \geq 0$, $r \in [0,\, 1)$. Thus, φ_r is a convex fonction on $[0,\, \infty)$ and φ_r' is increasing on $[0,\, \infty)$ for all $r \in [0,\, 1)$. Moreover,

$$\varphi_r'(0) = \frac{1}{4}\int_0^{2\pi} \ln\left(1+r^2-2r\cos s\right)\mathrm{d}s,$$

but $\psi(r) = \int_0^\pi \ln(1 + r^2 - 2r\cos s)\,\mathrm{d}s$ satisfies $2\psi(r) = \psi(r^2)$ for every $r \in [0,\,1)$, which gives by induction $\psi(r) = \frac{1}{2^k}\psi\left(r^{2^k}\right)$, for every $k = 0,\,1,\,2,\,\ldots$ As a consequence, taking the limit as k tends to $+\infty$ and using the continuity of ψ at point 0, we get $\psi(r) = 0$, for every $r \in [0,\,1)$. Moreover,

$$\int_\pi^{2\pi} \ln\left(1 + r^2 - 2r\cos s\right)\mathrm{d}s = -\int_0^{-\pi} \ln\left(1 + r^2 - 2r\cos(\pi - u)\right)\mathrm{d}u =$$

$$= \int_{-\pi}^0 \ln\left(1 + r^2 + 2r\cos(u)\right)\mathrm{d}u = \int_0^\pi \ln\left(1 + r^2 + 2r\cos(v + \pi)\right)\mathrm{d}v =$$

$$= \int_0^\pi \ln\left(1 + r^2 - 2r\cos(v)\right)\mathrm{d}v = \psi(r) = 0.$$

We get,

$$\varphi_r'(t) \geq \varphi_r'(0) = 0,\ \forall t \in [0,\,\infty),\ r \in [0,\,1),$$

and φ_r is increasing on $[0,\,\infty)$. The fact that

$$r \mapsto I(t,\,r),$$

is increasing on $[0,\,1)$, for all $t \geq 0$ is obvious since

$$I(t,\,r) = \left\|\frac{1}{(1 - rz)^{t/2}}\right\|_{H^2}^2 = \sum_{k\geq 0} a_k(t)^2 r^{2k},$$

where $a_k(t)$ is the k^{th} Taylor coefficient of $(1 - z)^{-t/2}$. The same reasoning gives that $r \mapsto \varphi_r(t)$ is increasing on $[0,\,1)$. ☐

Lemma 2. *If for some $r_0 \in [0,\,1)$ and $\gamma < t$, the function $\frac{w(\rho)}{(1-\rho^2)^\gamma}$ is increasing on $[r_0,\,1)$, then*

$$\int_r^1 \rho w(\rho) I(t,\,r\rho)\mathrm{d}\rho \asymp \int_{r_0}^1 \rho w(\rho) I(t,\,r\rho)\mathrm{d}\rho,$$

for all t such that $t - \gamma > 2$, and for all $r \geq r_0$, with constants independant on t.

Proof. Clearly,

$$\int_{r_0}^1 \rho w(\rho) I(t,\,r\rho)\mathrm{d}\rho \geq \int_r^1 \rho w(\rho) I(t,\,r\rho)\mathrm{d}\rho.$$

Moreover,

$$\int_{r_0}^{1} \rho w(\rho) I(t,\, r\rho)\mathrm{d}\rho = \int_{r}^{1} \rho w(\rho) I(t,\, r\rho)\mathrm{d}\rho + \int_{r_0}^{r} \rho w(\rho) I(t,\, r\rho)\mathrm{d}\rho,$$

and

$$\int_{r_0}^{r} \rho w(\rho) I(t,\, r\rho)\mathrm{d}\rho =$$

$$= \int_{r_0}^{r} \frac{\rho w(\rho)}{(1-\rho^2)^{\gamma}} \frac{(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho)\mathrm{d}\rho \leq$$

$$\leq \frac{w(r)}{(1-r^2)^{\gamma}} \int_{r_0}^{r} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho)\mathrm{d}\rho \leq$$

($u \mapsto J(t,\, u)$ is increasing for all $t > 0$)

$$\leq \frac{w(r)}{(1-r^2)^{\gamma}} J(t,\, r^2) \int_{r_0}^{r} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho.$$

On the other hand,

$$\int_{r}^{1} \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho)\mathrm{d}\rho =$$

$$= \int_{r}^{1} \frac{w(\rho)}{(1-\rho^2)^{\gamma}} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho)\mathrm{d}\rho \geq$$

($u \mapsto J(t,\, u)$ is increasing for all $t > 0$)

$$\geq \frac{w(r)}{(1-r^2)^{\gamma}} J(t,\, r^2) \int_{r}^{1} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho,$$

but

$$\int_{r}^{1} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho \asymp \int_{r_0}^{1} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho,$$

with constants independant on t since$t - \gamma > 2$. Thus, we obtain

$$\int_{r_0}^{r} \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho)\mathrm{d}\rho \leq \frac{w(r)}{(1-r^2)^{\gamma}} J(t,\, r^2) \int_{r_0}^{r} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho \leq$$

$$\leq \frac{w(r)}{(1-r^2)^{\gamma}} J(t,\, r^2) \int_{r_0}^{1} \frac{\rho\,(1-\rho^2)^{\gamma}}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho \leq$$

$$\leq Const. \frac{w(r)}{(1-r^2)^\gamma} J(t,\, r^2) \int_r^1 \frac{\rho\,(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} \mathrm{d}\rho \leq$$

$$\leq Const. \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} J(t,\, r\rho) \mathrm{d}\rho,$$

(where $Const$ is a constant which does not depend on t), which completes the proof. $\square$

Proof of statement (2) of the theorem (the lower bound only). For the minoration (with the same function $f(z) = \frac{1}{(1-rz)^n}$), using (8) with $\alpha = r_0$, we need to prove

$$\int_{r_0}^1 \rho w(\rho) I(pn+p,\, r\rho) \mathrm{d}\rho \geq \frac{C}{(1-r)^p} \int_{r_0}^1 \rho w(\rho) I(pn+p,\, r\rho) \mathrm{d}\rho,$$

which means (supposing that $r \geq r_0$) with our second lemma that,

$$\int_r^1 \rho w(\rho) I(pn+p,\, r\rho) \mathrm{d}\rho \geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) I(pn,\, r\rho) \mathrm{d}\rho,$$

which means with our first lemma that,

$$\int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{pn+p-1}} J(pn+p,\, r\rho) \mathrm{d}\rho \geq$$

$$\geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{pn-1}} J(pn,\, r\rho) \mathrm{d}\rho.$$

The last statement is obvious since

$$\int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{pn+p-1}} J(pn+p,\, r\rho) \mathrm{d}\rho \geq$$

$$\geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{pn-1}} J(pn+p,\, r\rho) \mathrm{d}\rho \geq$$

($t \mapsto J(t,\, u)$ is increasing for all $0 \leq u < 1$)

$$\geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{pn-1}} J(pn,\, r\rho) \mathrm{d}\rho =$$

$$= \frac{1}{(1-r^2)^p}\int_r^1 \rho w(\rho) I(pn,\, r\rho)\mathrm{d}\rho. \qquad \square$$

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